A SIMPLER KERNEL FOR STRATIFIED MUKAI FLOPS

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Abstract. We reinvestigate the problem of describing the Fourier-Mukai kernel for the derived equivalence associated to a stratified Mukai flop. For the case of Grassmannians of planes we give a very simple geometric construction of the kernel, using the framework of matrix factorizations.

Contents

1. Introduction

There is a famous conjecture of Bondal-Orlov [BO] and Kawamata [Kaw] that if two smooth varieties X_+ and X_- are connected by a *flop* then there should also be a derived equivalence:

$$D^b(X_+) \cong D^b(X_-)$$

Here we are using the word 'flop' as a synonym for 'K-equivalence', *i.e.* a birational transformation admitting a proper birational roof

$$X_+ \xleftarrow{\pi_+} \widetilde{X} \xrightarrow{\pi_-} X_-$$

with the property that $\pi_+^* K_{X_+} = \pi_-^* K_{X_-}$. The general case of this conjecture seems some way off, but it has been established in many classes of examples [Br, N1, CKL, S1, DS, S2, HLS, H, HL2, MSYZ] using a variety of different techniques.

Some of these techniques provide an explicit Fourier-Mukai kernel

$$\mathcal{K} \in D^b(X_+ \times X_-)$$

for the derived equivalence, but some do not. In the latter case, even once the equivalence has been proven, there is the follow-up problem of finding a description of the kernel. And this second problem is important; if one could formulate a plausible guess for $\mathcal{K}$ in general then that would provide an entry point for proving the conjecture.

One guess for $\mathcal{K}$ is to take the structure sheaf on the fibre product $X_+ \times_{X_0} X_-$ where X_0 is the common contraction of $X_\pm$. This works for standard flops [BO] and

Grassmannian flops [B+], where the fibre product is smooth and indeed the same as the obvious correspondence $\widetilde{X}$. It also works for Mukai flops [N1] and Abuaf flops [H], where the fibre product is reducible with two components. Unfortunately this guess fails once we get to *stratified Mukai flops*

$$X_+ = T^\vee \operatorname{Gr}(k,n) \dashleftarrow\dashrightarrow T^\vee \operatorname{Gr}(n,k) = X_-$$

where the fibre product has many irreducible components [N2].[1]

Nevertheless, it is known that stratified Mukai flops do induce derived equivalences. This was first proven in the remarkable paper of Cautis-Kamnitzer-Licata [CKL] as a consequence of of a much larger structure: a categorical $\mathfrak{sl}_2$ action. A follow-up paper [C] provided an explicit description of the kernel $\mathcal{K}$. It is a sheaf on the fibre product but not simply the structure sheaf; one must put specific line bundles on each component, away from the triple intersections, and then take a reflexive extension.[2]

A very different proof of the derived equivalence for stratified Mukai flops was found by Halpern-Leistner and Sam [HLS], as a special case of a large class of examples. There are two main ingredients in their approach:

(1) Building on work of Špenko-Van den Bergh [SVdB], they use the technique of 'magic windows' to prove that flops coming from quasi-symmetric GIT problems always give derived equivalences.
(2) Using Knörrer periodicity they replace the derived categories in the stratified Mukai flop with categories of matrix factorizations on two higher-dimensional spaces. These higher-dimensional spaces are related by a flop of the form considered in (1).

Recent work of the second author [T] shows that these two proofs produce the *same* set of derived equivalences.[3]

The aim of the current paper is to unpack the Halpern-Leistner–Sam equivalence and provide an explicit description of the kernel, as an object in the relevant category of matrix factorizations. The bad news is that we only manage to solve the problem in the case $k = 2$. The good news is that, in that case, the answer is as simple as one could hope for. In particular it is the fibre product of the higher-dimensional spaces. We explain our result in Section 1.1 below.

Our conclusion is that the complexity of stratified Muki flops is (at least for $k = 2$) an artifact of Knörrer periodicity; if one is happy to stay in the world of matrix factorizations then the kernel is easy to construct, and the intracies found by [C] arise only when one passes back to the world of derived categories.

Of course it would be nice to be able to extend our results to $k > 2$. Unfortunately part of our argument (Section 4.3) is very specific to the $k = 2$ case and it is unclear to us what the correct generalization should be.

1.1. **Results and outline.** Let V be a finite-dimensional vector space, choose $0 < k < \dim V$, and let S denote the tautological rank k bundle on $\operatorname{Gr}(k, V)$. The

[1]It may be tempting to try to repair this using the derived fibre product, but this fails already for standard flops. There the derived and classical fibre products are different and it is the classical one that gives the equivalence.
[2]The special case where $(k, n) = (2, 4)$ had been worked out earlier by Kawamata [Kaw2].
[3]*I.e.* they give the same equivalence, up to shifts and the Picard group of both sides. One can make a physics argument that predicts this statement by examining the topology of the relevant stringy Kähler moduli space.

cotangent bundle of the Grassmannian is:

$$X_+ = T^\vee \operatorname{Gr}(k,V) = \operatorname{Hom}(V/S,S)_{\operatorname{Gr}(k,V)}$$

It sits as a subbundle inside $\operatorname{Hom}(V,S)_{\operatorname{Gr}(k,V)}$, where it is cut out by a transverse section of the pull up of the bundle $\operatorname{End}(S)$. If we write x and y for the tautological sections of $\operatorname{Hom}(S,V)$ and $\operatorname{Hom}(V,S)$ then the section cutting out X_+ is:

$$y \circ x \ \in \Gamma(\operatorname{End}(S))$$

Now let Q denote the tautological rank k quotient bundle on the Grassmannian of quotients $\operatorname{Gr}(V,k)$. In a similar way we have a subbundle:

$$X_- = T^\vee \operatorname{Gr}(V,k) \ \hookrightarrow\ \operatorname{Hom}(Q,V)_{\operatorname{Gr}(V,k)}$$

The *Grassmannian flop* is the evident birational equivalence between the spaces $\operatorname{Hom}(V,S)_{\operatorname{Gr}(k,V)}$ and $\operatorname{Hom}(Q,V)_{\operatorname{Gr}(V,k)}$. It induces a birational equivalence between X_+ and X_-, this is the *stratified Mukai flop*.

Now write

$$E_+ = \operatorname{Hom}(V,S) \oplus \operatorname{End}(S)_{\operatorname{Gr}(k,V)}$$

for the total space of the bundle $\operatorname{End}(S)$, pulled-up to $\operatorname{Hom}(V,S)_{\operatorname{Gr}(k,V)}$. The section $y \circ x$ becomes (since $\operatorname{End}(S)$ is self-dual) a function on this total space E_+, which we can write as

$$W = \operatorname{tr}(p \circ y \circ x) \quad \in \Gamma(\mathcal{O}_{E_+})$$

where p is the tautological section of $\operatorname{End}(S)$. By Knörrer periodicity [O, Sh, ...] there is an equivalence

$$D^b(X_+) \ \cong\ \operatorname{MF}(E_+, W)$$

from the derived category of X_+ to the category of matrix factorizations of W. Similarly, $D^b(X_-)$ is equivalent to the category of matrix factorizations of a function W' on:

$$E_- = \operatorname{Hom}(Q,V) \oplus \operatorname{End}(Q)_{\operatorname{Gr}(V,2)}$$

The spaces E_+ and E_- are also birational, and the equivalence between $D^b(X_+)$ and $D^b(X_-)$ can be viewed—via Knörrer periodicity—as coming from an equivalence:

$$\operatorname{MF}(E_+, W) \ \cong\ \operatorname{MF}(E_-, W')$$

In this description the Fourier Mukai kernel for the equivalence is an object of:

$$\operatorname{MF}(E_+ \times E_-,\ W' - W)$$

Theorem 1.1. *Set $k = 2$. Let $E^o \subset E_\pm$ be the open subset where E_+ and E_- are isomorphic and write*

$$\Delta \ \subset\ E^o \times E^o \ \subset\ E_+ \times E_-$$

for the diagonal in this subset. Then the structure sheaf on the closure of Δ

$$\mathcal{O}_{\overline{\Delta}} \in \operatorname{MF}(E_+ \times E_-,\ W' - W)$$

is the kernel for the window equivalence of [HLS].

Remark 1.2.
(1) The functions W and W' agree on E^o, so $W' - W$ vanishes on $\overline{\Delta}$, hence $\mathcal{O}_{\overline{\Delta}}$ is indeed an object in the matrix factorization category.
(2) The equivalence we are discussing restricts to the identity on E^o, so the kernel must *a priori* be some extension of the sheaf $\mathcal{O}_\Delta$. Hence our result really is the simplest we could hope for.

(3) Write E_0 for the affine singularity underlying $E_\pm$. The variety $\overline{\Delta}$ agrees with the fibre product:

$$\overline{\Delta} = E_+ \times_{E_0} E_-$$

It is an irreducible and singular variety which can be described as a cone over $\mathrm{Gr}(2, V) \times \mathrm{Gr}(V, 2)$. It carries a function which is the restriction of either W or W', and we have a correspondence of LG models:

$$\begin{tikzcd} & (\overline{\Delta}, W) \arrow[dl] \arrow[dr] & \\ (E_+, W) & & (E_-, W') \end{tikzcd}$$

Another way to say Theorem 1.1 is that the window equivalence is induced by this correspondence.

We will explain the geometry of $\overline{\Delta}$ in Section 4.1, and the proof of Theorem 1.1 in the remainder of Section 4.

Via Knörrer perodicity $\mathcal{O}_{\overline{\Delta}}$ is equivalent to some object $\mathcal{K}$ in $D^b(X_+ \times X_-)$. By the results of [T], we know that this $\mathcal{K}$ must be the Fourier-Mukai kernel found by Cautis in [C]. For the sake of concreteness we compare them explicitly in Section 5.

1.2. **Acknowledgements.** Although we do not formally rely on their results we were heavily inspired by the papers [BDF] and [B+] that discuss the problem of producing kernels for window equivalences. After completing this paper we learned from Matt Ballard that they have independently obtained our results for $k = 2$, and also solved the $k = 3$ case with some computer algebra [BCF].

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2. Magic windows and kernels

Let G be a reductive group and U a representation of G which is self-dual.[4] Write $\mathcal{Y}$ for the Artin stack:

$$\mathcal{Y} = [U/G]$$

Given an object $\mathcal{E} \in D^b(\mathcal{Y})$, its restriction $\mathcal{E}|_0$ to the origin lies in $D^b(\mathrm{B}G)$, so is a direct sum of shifts of G-representations. For a given finite set $\mathfrak{W}$ of irreducible representations of G we define the *grade-restricted subcategory*, or *window*, to be the full subcategory:

$$\mathcal{M}_{\mathfrak{W}} = \left\{ \mathcal{E} \in D^b(\mathcal{Y}),\ \mathcal{E}|_0 \text{ contains only irreps from } \mathfrak{W} \right\} \subset D^b(\mathcal{Y})$$

In [HLS] (following [SVdB]) certain specific subsets $\mathfrak{W}$ are defined, depending on the represention U, with a remarkable property.[5] The corresponding subcategories $\mathcal{M}_{\mathfrak{W}}$ are called *magic windows.*

Theorem 2.1. [HLS, Thm. 3.2] *Let $Y = U /\!\!/_\theta G$ be a GIT quotient of U for a generic stability condition θ. Then the restriction functor*

$$\mathcal{M}_{\mathfrak{W}} \to D^b(Y)$$

is an equivalence.

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[4] Or more generally quasi-symmetric.

[5] We will not give the combinatorial details of how these sets $\mathfrak{W}$ are defined but we will say explicitly what they are in the examples we care about.

Here the GIT quotient Y means, by definition, the open substack of $\mathcal{Y}$ consisting of θ-semi-stable points. So there is a restriction functor $D^b(\mathcal{Y}) \to D^b(Y)$, which we are evaluating on the subcategory $\mathcal{M}_{\mathfrak{W}}$.

One of the key points of the above thorem is that $\mathfrak{W}$ only depends on U and not on θ. The same window category works for all (generic) GIT quotients. Consequently if Y_+ and Y_- are two different GIT quotients then we get a derived equivalence:

$$D^b(Y_+) \xleftarrow{\sim} \mathcal{M}_{\mathfrak{W}} \xrightarrow{\sim} D^b(Y_-)$$

This is called a *window equivalence*.

Remark 2.2. Associated to $\mathfrak{W}$ there is a finite set of vector bundles on $\mathcal{Y}$ which obviously lie in $\mathcal{M}_{\mathfrak{W}}$. In fact these vector bundles generate $\mathcal{M}_{\mathfrak{W}}$, and this is a large part of the proof of the theorem.

Now we discuss the kernels for these window equivalences. Firstly, from Theorem 2.1 we get an embedding functor

$$D^b(Y_+) \hookrightarrow D^b(\mathcal{Y})$$

whose image is $\mathcal{M}_{\mathfrak{W}}$. This functor has some kernel $\mathcal{K} \in D^b(Y_+ \times \mathcal{Y})$ with the following two properties:

(1) The restriction of $\mathcal{K}$ to $Y_+ \times Y_+$ is the structure sheaf $\mathcal{O}_\Delta$ on the diagonal, because the composition of the embedding functor with restriction is the identity on $D^b(Y_+)$.

(2) Consider the object:

$$\mathcal{K}|_{Y_+ \times \{0\}} \in D^b(Y_+ \times BG)$$

It has a weight decomposition, *i.e.* it splits as a direct sum of objects in $D^b(Y_+)$ tensored with irreducible G-representations. Since the image of the embedding functor is $\mathcal{M}_{\mathfrak{W}}$, only irreducible representations from $\mathfrak{W}$ can occur as weights of $\mathcal{K}|_{Y_+ \times \{0\}}$.

A consequence of Theorem 2.1 is that $\mathcal{K}$ is the *unique* object with these two properties; see [HL1, Sect. 2.3].[6] And once we have $\mathcal{K}$ the kernel for the window equivalence is easy; it's simply the restriction of $\mathcal{K}$ to the open substack $Y_+ \times Y_-$.

So the problem is to find the object $\mathcal{K}$. In principle there is an algorithm for doing this: choose an arbitrary extension $\mathcal{K}'$ of $\mathcal{O}_\Delta$, then there is a specific sequence of mutations which will bring it into the required window, without affecting property (1). However in practice this is rarely feasible. A better approach is to:

- Make the correct guess for $\mathcal{K}$ first time.
- Prove that your guess is correct by computing the weights of $\mathcal{K}|_{Y_+ \times \{0\}}$.

This is the approach that we use in this paper.

Remark 2.3. Suppose we have a G-invariant function W on U. This defines a function on $\mathcal{Y}$ and (by restriction) a function on any GIT quotient. We can replace derived categories with categories of matrix factorizations of W in all the above, Theorem 2.1 remains true, and the story is unchanged. The function W is irrelevant for the problem of finding $\mathcal{K}$.

[6] Roughly, the argument is that $D^b(Y_+ \times Y_+)$ is equivalent to the subcategory $D^b(Y_+) \boxtimes \mathcal{M}_{\mathfrak{W}}$ in $D^b(Y_+ \times \mathcal{Y})$, and $\mathcal{K}$ is the object corresponding to $\mathcal{O}_\Delta$.

3. Review of the Grassmannian flop

Fix vector spaces S, V of dimensions 2 and $n > 2$, and consider the Artin stack:

$$\mathcal{F} \;=\; [\,\mathrm{Hom}(S,V) \oplus \mathrm{Hom}(V,S) \;/\; \mathrm{GL}(S)\,]$$

There are two generic GIT quotients of $\mathcal{F}$, associated to either positive or negative powers of the character $\det(S)$, they are:

$$F_+ = \mathrm{Hom}(V,S)_{Gr(2,V)} \qquad \text{and} \qquad F_- = \mathrm{Hom}(S,V)_{\mathrm{Gr}(V,2)}$$

This is the $k = 2$ case of the Grassmannian flop. Write F_0 for the affine singularity that underlies them both, it is the subvariety

$$F_0 \;\subset\; \mathrm{End}(V)$$

of endomorphisms of rank ≤ 2. Also write $F^o \subset F_\pm$ for the open set where the two spaces are isomorphic. In this example three spaces coincide:

(1) The obvious correspondence

$$\widetilde{F} \;= \mathrm{Hom}(Q,S)_{\mathrm{Gr}(2,V)\times \mathrm{Gr}(V,2)}$$

between F_+ and F_-.

(2) The fibre product $F_+ \times_{F_0} F_-$.

(3) The closure

$$\overline{\Delta_F} \;\subset\; F_+ \times F_-$$

where $\Delta_F \subset F^o \times F^o$ is the diagonal. In $\widetilde{F}$, the locus Δ_F is where the map from Q to S is an isomorphism.

The structure sheaf on $\tilde{F}$ induces a derived equivalence between F_+ and F_- [DS, BLV, B+]. Let us recall the proof of this in the 'magic window' framework from Section 2.

In this example, a magic window can be constructed from the following set of irreducible representations of the group $\mathrm{GL}(S) \cong \mathrm{GL}_2$:

$$\mathfrak{W} = \left\{ \mathrm{Sym}^k S \otimes (\det S)^l,\; l \geq 0,\; k + l < n - 1 \right\} \tag{3.1}$$

These are the irreducible representations corresponding to Kapranov's exceptional collection on $\mathrm{Gr}(2, n)$ [Kap]. Now we consider the further closure:

$$\overline{\overline{\Delta_F}} \;\subset F_+ \times \mathcal{F}$$

The claim is that the structure sheaf on this locus is the kernel for the embedding functor

$$D^b(F_+) \xrightarrow{\sim} \mathcal{M}_{\mathfrak{W}} \subset D^b(\mathcal{F})$$

which implies immediately that $\mathcal{O}_{\overline{\overline{\Delta_F}}}$ is the kernel for the window equivalence. And as explained in the previous section, to prove this claim we just need to verify:

Lemma 3.2. [B+, Kap] *The restriction of $\mathcal{O}_{\overline{\overline{\Delta_F}}}$ to $F_+ \times \{0\}$ has weights contained in the set $\mathfrak{W}$.*

Proof. We introduce co-ordinates on $F_+ \times \mathcal{F}$ as follows

$$S_1 \underset{y_1}{\overset{x_1}{\rightleftarrows}} V \underset{x_2}{\overset{y_2}{\rightleftarrows}} S_2$$

where x_1 is an injection. Now consider the stack

$$\mathcal{D}_F \;\;\subset\;\; [\,\mathrm{Hom}(S_1,V) \oplus \mathrm{Hom}(V,S_2) \oplus \mathrm{Hom}(S_2,S_1) \;/\; \mathrm{GL}(S_1) \times \mathrm{GL}(S_2)\,]$$

where the first map is an injection. We give $\mathcal{D}_F$ co-ordinates:

$$\begin{tikzcd} S_1 \arrow[r, hook, "x_1"] & V \arrow[r, "y_2"] & S_2 \arrow[ll, bend left, "a"] \end{tikzcd}$$

There is an obvious map $\mathcal{D}_F \to F_+ \times \mathcal{F}$ by setting $y_1 = ay_2$ and $x_2 = x_1 a$, and this map is an embedding. Note that $\overline{\Delta_F} = \widetilde{F}$ is the locus in $\mathcal{D}_F$ where y_2 is a surjection, and Δ_F is where additionally a is an isomorphism.

Now factor the map $\mathcal{D}_F \to F_+ \times \mathcal{F}$ through the intermediate stack $\mathcal{H}_F$ which has co-ordinates:

$$\begin{tikzcd} S_1 \arrow[r, hook, "x_1"] & V \arrow[r, shift left, "y_2"] & S_2 \arrow[l, hook', shift left, "x_2"] \arrow[ll, bend left, "a"] \end{tikzcd} \tag{3.3}$$

This stack $\mathcal{H}_F$ is flat over $\mathcal{F}$, and $\mathcal{D}_F \subset \mathcal{H}_F$ is cut out by the transverse section:

$$x_1 a - x_2 \ \in \Gamma\big(\mathcal{H}_F, \mathrm{Hom}(S_2, V)\big)$$

So we can resolve $\mathcal{O}_{\mathcal{D}_F} = \mathcal{O}_{\overline{\Delta_F}}$ with the Koszul complex of this section. Restricting to $F_+ \times \{0\}$, we get the Kozul complex of the section $x_1 a \in \Gamma(\mathrm{Hom}(S_2, V))$ on $F_+ \times \mathrm{BGL}(S_2)$.

This section $x_1 a$ is not transverse, but it is a transverse section of the subbundle $\mathrm{Hom}(S_2, S_1)$. So the homology of the Kozsul complex is the exterior algebra on the dual of the quotient bundle:

$$\mathrm{Hom}(S_2, V/S_1)^\vee$$

The $\mathrm{GL}(S_2)$-irreps that occur in this exterior algebra are precisely the set $\mathfrak{W}$. □

Remark 3.4. The same argument works for $k > 2$, replacing $\mathfrak{W}$ with the set of Schur powers $\mathbb{S}^\gamma S$ indexed by Young diagrams with $\mathrm{width}(\gamma) \leq n - k$.

4. Stratified Mukai flops

4.1. **The geometry of the kernel.** For the stratified Mukai flop, we introduce the stack

$$\mathcal{E} \ = \ [\ \mathrm{Hom}(S, V) \oplus \mathrm{Hom}(V, S) \oplus \mathrm{End}(S) \ / \ \mathrm{GL}(S)\ ]$$

where as before $\dim(S) = 2$ and $\dim(V) = n > 2$. We use co-ordinates x, y and p for the three summands. This stack contains the spaces $E_\pm$ from Section 1.1 as the open substacks

$$E_+ = \{\mathrm{rank}(x) = 2\} \qquad \text{and} \qquad E_- = \{\mathrm{rank}(y) = 2\}$$

although these are not quite the GIT quotients; see Remark 4.5. Write E_0 for the affine singularity underlying these three spaces.

Lemma 4.1. *E_0 embeds into the vector space*

$$\mathrm{End}(V)^{\oplus 2} \oplus \mathbb{C}^2$$

via the map:

$$(x, y, p) \ \mapsto \ \big(xy,\, xpy,\, \det(p),\, \mathrm{tr}(p)\big)$$

Proof. The claim is about the generators of the ring of $\mathrm{GL}(S)$-invariant functions on $\mathcal{E}$. Firstly, by picking a basis for V and scaling the rows of x and the columns of y separately we obtain a $(\mathbb{C}^*)^{2n}$ action on $\mathcal{E}$, commuting with the $\mathrm{GL}(S)$ action, *i.e.* a $2n$-dimensional multigrading on the ring of functions. By exploiting this multigrading as in [KP, §4] we can essentially reduce the problem to the case $n = 1$.

Next, using the diagonal 1-parameter subgroup in $\mathrm{GL}(S)$ it's clear that any invariant function must be a polynomial in the entries of p and the entries of the rank 1 matrix:

$$q = yx \quad \in \mathrm{End}(S)$$

Then by [KP, p. 21], the Cayley-Hamilton theorem, and the fact that $\det(q) = 0$, the ring of invariants (for $n = 1$) is generated by:

$$\det(p), \qquad \mathrm{tr}(p), \qquad \mathrm{tr}(q) = xy \qquad \text{and} \qquad \mathrm{tr}(pq) = xpy$$

Returning to the case of general n, the invariants are generated by $\det(p), \mathrm{tr}(p)$ and the individual entries of xy and xpy. $\square$

Now we want to understand the closure of the diagonal. Recall the stack $\mathcal{D}_F$ from Lemma 3.2, and let $\mathcal{D}$ be the total space of the vector bundle $\mathrm{End}(S_1) \oplus \mathrm{End}(S_2)$ over $\mathcal{D}_F$. Thus $\mathcal{D}$ has co-ordinates

$$\begin{tikzcd} S_1 \arrow[loop left, "p_1"] \arrow[r, hook, "x_1"] & V \arrow[r, "y_2"] & S_2 \arrow[loop right, "p_2"] \arrow[ll, bend left, "a"] \end{tikzcd}$$

(where x_1 is an injection) and there is an embedding $\mathcal{D} \hookrightarrow E_+ \times \mathcal{E}$. Recall that $E^o \subset E_\pm$ denotes the common open subset, and $\Delta \subset E^o \times E^o$ the diagonal. Then Δ is a subset of $\mathcal{D}$, it is the locus where y_2 and a have full rank and also:

$$p_2 = a^{-1} p_1 a$$

We can take the closure $\overline{\Delta} \subset E_+ \times E_-$, or the further closure $\overline{\overline{\Delta}} \subset E_+ \times \mathcal{E}$; both lie inside $\mathcal{D}$. In fact $\overline{\overline{\Delta}}$ is the closed substack:

$$\overline{\overline{\Delta}} = \{\ a p_2 = p_1 a,\ \det(p_1) = \det(p_2),\ \mathrm{tr}(p_1) = \mathrm{tr}(p_2)\ \} \ \subset \mathcal{D} \tag{4.2}$$

Note that the first equation alone is not enough, it cuts out a reducible subvariety with three equidimensional components (indexed by the rank of a). We need the remaining equations to specify the correct irreducible component. In particular $\overline{\overline{\Delta}}$ is *not* a complete intersection.

It follows from Lemma 4.1 that $\overline{\Delta}$ agrees with the fibre product $E_+ \times_{E_0} E_-$. It is a quadric cone over $\mathrm{Gr}(2, n) \times \mathrm{Gr}(n, 2)$.

4.2. **The weights of the kernel.** Now we introduce, as in Section 1.1, the invariant function

$$W = \mathrm{tr}(pyx) \quad \in \Gamma(\mathcal{O}_\mathcal{E})$$

and the corresponding category of matrix factorizations $\mathrm{MF}(\mathcal{E}, W)$. By Knörrer periodicity $\mathrm{MF}(\mathcal{E}, W)$ is equivalent to the derived category of the zero locus:

$$\{yx = 0\} \quad \subset \mathcal{F}$$

The two sides of the statified Mukai flop, $X_\pm$, are the GIT quotients of this stack. This is hyperkähler reduction; the equation $yx = 0$ is the complex moment map for the action of $\mathrm{GL}(S)$ on the complex symplectic space $\mathrm{Hom}(S, V) \oplus \mathrm{Hom}(V, S)$.

Remark 4.3. We also include an additional $\mathbb{C}^*$ action on $\mathcal{E}$ (the 'R-charge') which acts on (x, y, p) with weights $(0, 0, 2)$ and hence on W with weight 2. This is required for the category of matrix factorizations to be $\mathbb{Z}$-graded and for Knörrer periodicity to work as stated. But it is essentially irrelevant for our proof of Theorem 1.1 so we will avoid mentioning it as far as possible.

For the stack $\mathcal{E}$ a magic window can be built from the following set of $\mathrm{GL}(S)$-irreps:

$$\mathfrak{W}' = \left\{ \mathrm{Sym}^k S \otimes (\det S)^l,\ l \geq 0,\ k + l < n \right\} \tag{4.4}$$

Note the difference between this and the set $\mathfrak{W}$ for the Grassmannian flop (3.1). The additional factor of $\mathrm{End}(S)$ relaxes the bound on $k + l$ by 1.

By Theorem 2.1 the corresponding magic window $\mathcal{M}_{\mathfrak{W}'} \subset D^b(\mathcal{E})$ is equivalent to the derived category of either GIT quotient. There is also a window in the matrix factorization category

$$\mathcal{M}^W_{\mathfrak{W}'} \subset \mathrm{MF}(\mathcal{E}, W)$$

defined by exactly the same grade-restriction rule as $\mathcal{M}_{\mathfrak{W}'}$. Again the restriction functors from $\mathcal{M}^W_{\mathfrak{W}'}$ to matrix factorizations on either GIT quotient are equivalences [HLS, Thm. 5.1]

Remark 4.5. The GIT quotients of $\mathcal{E}$ are not $E_\pm$ but slightly larger open substacks. The reason is that points where $\operatorname{rank} x < 2$ (or $\operatorname{rank} y < 2$) can be semi-stable, provided some condition on p holds. However, the critical locus of W is contained entirely in $E_\pm$, so these additional semi-stable points do not affect the category of matrix factorizations. Hence the restriction functors

$$\mathcal{M}^W_{\mathfrak{W}'} \to \mathrm{MF}(E_\pm, W)$$

are both equivalences. This is a general fact about hyperkähler quotients of this form [HLS, §5].

Now we consider the stack $E_+ \times \mathcal{E}$ with the obvious function $W_2 - W_1$. The sky-scraper sheaf $\mathcal{O}_{\overline{\overline{\Delta}}}$ defines an object in the category

$$\mathrm{MF}\big(E_+ \times \mathcal{E},\, W_2 - W_1\big)$$

since $\overline{\overline{\Delta}}$ is contained in the locus where $W_1 = W_2$. Theorem 1.1 follows immediately from the following:

Proposition 4.6. *The restriction of $\mathcal{O}_{\overline{\overline{\Delta}}}$ to $E_+ \times \{0\}$ has weights contained in the set $\mathfrak{W}'$.*

Proof. Recall from Section 4.1 the stack $\mathcal{D}$, which is the total space of the vector bundle $\mathrm{End}(S_1) \oplus \mathrm{End}(S_2)$ over the stack $\mathcal{D}_F$ that we used in Lemma 3.2. Write $\mathcal{H}$ for the total space of the same vector bundle over the stack $\mathcal{H}_F$ (3.3). Then we have inclusions

$$\overline{\overline{\Delta}} \hookrightarrow \mathcal{D} \hookrightarrow \mathcal{H} \hookrightarrow E_+ \times \mathcal{E}$$

and $\mathcal{H}$ is flat over $\mathcal{E}$. Moreover, by (4.2) the locus $\overline{\overline{\Delta}}$ is the intersection of $\mathcal{D}$ with the substack:

$$\mathcal{C} = \{\, ap_2 = p_1 a,\ \det(p_1) = \det(p_2),\ \ \mathrm{tr}(p_1) = \mathrm{tr}(p_2)\}\ \subset \mathcal{H} \tag{4.7}$$

Since $\mathcal{D} \subset \mathcal{H}$ is the subset $\{x_1 a = x_2\}$ the intersection of $\mathcal{C}$ and $\mathcal{D}$ is obviously transverse. Hence

$$\mathcal{O}_{\overline{\overline{\Delta}}} \;=\; \mathcal{O}_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{D}}$$

does not need to be derived, and the weights of $\mathcal{O}_{\overline{\overline{\Delta}}}$ along $E_+ \times \{0\}$ are just the tensor products of the weights of $\mathcal{O}_{\mathcal{C}}$ and $\mathcal{O}_{\mathcal{D}}$.

The stack $E_+ \times \mathcal{E}$ is a vector bundle over $F_+ \times \mathcal{F}$ and $\mathcal{D}$ and $\mathcal{H}$ are just the restrictions of this bundle to the substacks $\mathcal{D}_F$ and $\mathcal{H}_F$. This means that computing the weights of $\mathcal{O}_{\mathcal{D}}$ is the same problem as we encountered in the Grassmannian flop, and the answer is the set $\mathfrak{W}$ (Lemma 3.2).

Thus the problem is to compute the weights of $\mathcal{O}_{\mathcal{C}}$, *i.e.* the additional weights that appear from the (non-transverse) equations for $\mathcal{C}$. To do this we view $\mathcal{H}$ as the vector bundle

$$\mathrm{Hom}(S_2, S_1) \oplus \mathrm{End}(S_1) \oplus \mathrm{End}(S_2)$$

over $\mathrm{Gr}(2, V) \times \mathcal{F}$. Then $\mathcal{C}$ is a cone in this vector bundle cut out by the equations (4.7). Furthermore we can ignore traces; we have splittings

$$\mathrm{End}(S_i) = \mathrm{End}_0(S_i) \oplus \mathbb{C}$$

where the first factor is the trace-free endomorphisms, and it is clear that we can replace $\mathcal{C}$ with the set

$$\{\, ap_2 = p_1 a,\ \det(p_1) = \det(p_2)\,\}\ \ \subset\ \mathrm{Hom}(S_2, S_1) \oplus \mathrm{End}_0(S_1) \oplus \mathrm{End}_0(S_2) \tag{4.8}$$

without affecting our weight computation. In Section 4.3 below we show how to complete this computation and find that the weights of $\mathcal{O}_{\mathcal{C}}$ are exactly:

$$\mathcal{O},\ S_2,\ \text{and}\ \det S_2$$

Tensoring $\mathfrak{W}$ by these irreducible representations gives precisely the set $\mathfrak{W}'$. $\square$

4.3. **A free resolution.** Let H be a 4-dimensional vector space. Consider the $\mathrm{GL}(H)$-representation

$$\wedge^2 H \oplus H$$

with co-ordinates (p, a), and let C be the invariant subvariety

$$C = \{\, p \wedge p = 0,\ p \wedge a = 0 \,\}$$

(the first equation is the Plücker relation). Thinking of this as a GIT problem, this C is precisely the subvariety destabilized by a 1-parameter subgroup λ of the form:

$$\gamma : \mathbb{C}^* \longrightarrow \mathrm{GL}(H), \quad t \longmapsto \begin{pmatrix} t^{-1} & & & \\ & t^{-1} & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

This means we have a Springer-type resolution of C given by the vector bundle

$$\widetilde{C} = (\det U \oplus U)_{\mathrm{Gr}(2,H)} \longrightarrow C$$

where U is the tautological subbundle on $\mathrm{Gr}(2, H)$. Hence, following Weyman [W], we can construct a free resolution of $\mathcal{O}_C$ by embedding $\widetilde{C}$ inside $\wedge^2 H \times H \times \mathrm{Gr}(2, H)$, taking its Koszul resolution, and then pushing down. It is straight-forward to compute that the resulting resolution is:

$$(\det H)^{-2} \longrightarrow H^\vee (\det H)^{-1} \oplus (\det H)^{-1} \longrightarrow (\det H)^{-1} \oplus \wedge^3 H^\vee \longrightarrow \mathcal{O}$$

Now take two 2-dimensional vector spaces S_1, S_2 and set $H = \mathrm{Hom}(S_2, S_1)$. We observe that we have an isomorphism of $\mathrm{GL}(S_1) \times \mathrm{GL}(S_2)$ representations

$$\mathrm{End}_0(S_1) \oplus \mathrm{End}_0(S_2) = \wedge^2 H \otimes (\det S_1)^{-1} (\det S_2) \tag{4.9}$$

where as above $\mathrm{End}_0(S_i)$ denotes the trace-free endomorphisms. So the representation $\wedge^2 H \oplus H$ is essentially the same as the representation (4.8). Moreover C corresponds to the locus

$$\{\, a p_2 = p_1 a,\ \det(p_1) = \det(p_2) \,\}$$

(this requires the correct choice of signs in the isomorphism (4.9)). So after modifying by the correct powers of $\det S_i$ we get a free resolution of $\mathcal{O}_C$ of the form:

$$(\det S_2)(\det S_1)^{-1} \longrightarrow \mathrm{Hom}(S_1, S_2) \oplus (\det S_2)(\det S_1)^{-1} \longrightarrow \mathcal{O} \oplus \mathrm{Hom}(S_1, S_2) \longrightarrow \mathcal{O} \tag{4.10}$$

Thus the restriction of $\mathcal{O}_C$ to the origin contains the $\mathrm{GL}(S_2)$-irreps

$$\mathbb{C},\ S_2,\ \det S_2$$

and no others.

Remark 4.11. For the purposes of Section 5 let us make a comment on shifts and R-charges (Remark 4.3). Implicit in the resolution (4.10) is the standard homological algebra convention that the degrees shift by 1 as we move right-to-left across the page, so the differentials all have degree 1. However, to work with matrix factorizations we must take into account R-charge, and then this convention is not quite correct. For example the first factor of the final map is:

$$\det(p_2) - \det(p_1) : \mathcal{O} \longrightarrow \mathcal{O}$$

But the R-charge of $\det(p_i)$ is 4, so for the differential to have degree 1 we must interpret it as a map:

$$\det(p_2) - \det(p_1) : \mathcal{O}[-3] \longrightarrow \mathcal{O}$$

Similar degree shifts apply to the other summands. If we take these into account, and use the standard position-on-the-page convention, then (4.10) should really be:

$$\begin{tikzcd}
\mathcal{O} & & & \mathcal{O} \arrow[lll, bend right] \\
\oplus & \mathrm{Hom}(S_1,S_2) \arrow[ul] & \mathrm{Hom}(S_1,S_2) \arrow[l] \arrow[ur] & \oplus \\
(\det S_2)(\det S_1)^{-1} \arrow[ur] & & & (\det S_2)(\det S_1)^{-1} \arrow[ul] \arrow[lll, bend left]
\end{tikzcd} \tag{4.12}$$

It is possible to turn this into a matrix factorization by adding additional arrows.

We repeat that this has no relevance to Theorem 1.1 since shifts are of no importance there, only $\mathrm{GL}(S_2)$ weights.

5. Comparison to CKL

We return temporarily to the case of general k and n. As before we let $X_\pm$ be the two sides of the stratified Mukai flop, with co-ordinates (x, y) satisfying the moment map condition $yx = 0$ (Section 4.2). Also recall the space $\widetilde{F} = \overline{\Delta}_F$ from Section 3 which has co-ordinates:

$$\begin{tikzcd}
S_1 \arrow[r, hook, "x_1"] & V \arrow[r, two heads, "y_2"] & S_2 \arrow[ll, bend left, "a"]
\end{tikzcd}$$

The fibre product $X_+ \times_{X_0} X_-$ is the subvariety of $\widetilde{F}$ cut out by the equations:

$$ay_2x_1 = 0 = y_2x_1a \tag{5.1}$$

It has $k+1$ irreducible components $Z_0, ...Z_k$, distinguished by specific rank conditions:

$$Z_i = \{\, (x_1, y_2, a),\ \mathrm{rank}(a) \leq k - i,\ \mathrm{null}(y_2x_1) \geq k - i\,\}$$

Let $\widetilde{Z}_i$ be the normalization of Z_i, and set Θ_i to be the shifted line bundle:

$$\Theta_i = \mathcal{O}_{\widetilde{Z}_i} \otimes (\det S_1)^i \otimes (\det S_2)^{-i}[-i].$$

In [CKL], Cautis, Kamnitzer and Licata constructed a *Rickard complex*

$$\Theta = \left\{ \Theta_k \xrightarrow{d} \cdots \xrightarrow{d} \Theta_1 \xrightarrow{d} \Theta_0 \right\} \tag{5.2}$$

as a geometric categorification of the braid group action of the quantized $\mathfrak{sl}_2$ on the K-theory of the 'quiver varieties' $\sqcup_k T^\vee \mathrm{Gr}(k, V)$.

Theorem 5.3 ([CKL]). *The complex (5.2) has a unique convolution $\mathcal{T} = \mathrm{Conv}(\Theta)$, which defines the Fourier-Mukai kernel of a derived equivalence:*

$$\Phi_{\mathcal{T}} : D^b(T^* \mathrm{Gr}(k, V)) \xrightarrow{\sim} D^b(T^* \mathrm{Gr}(V, k)).$$

In [T], the second author translated the Rickard complex (5.2) into the context of matrix factorizations. If we denote the Knörrer periodicity equivalence by

$$\Psi : D^b(X_\pm) \xrightarrow{\sim} \mathrm{MF}(E_\pm, W)$$

then it was shown ([T, Prop. 3.8]) that each functor $\Psi\Phi_{\Theta_i}\Psi^{-1}$ is induced by a matrix factorization kernel of the form:

$$(\pi_i)_*\mathcal{O}_{I_i} \otimes (\det S_1)^k \otimes (\det S_2)^{-k}[-i^2 - i] \tag{5.4}$$

Here $\pi_i : I_i \to E_+ \times E_-$ is a projection from a variety I_i which we describe as follows. First consider the flag variety $\mathrm{Fl}(k - i, k, n)$ with its two tautological bundles $S' \subset S_1 \subset V$. Now take the total space

$$\mathrm{End}(S') \oplus \mathrm{End}(S_1) \oplus \mathrm{End}(S_2) \longrightarrow \mathrm{Fl}(k - i, k, n) \times \mathrm{Gr}(n, k)$$

which we give co-ordinates

$$\begin{tikzcd}
S_1 \arrow[loop above, "p_1"] \arrow[r, hook, "x_1"] & V \arrow[r, two heads, "y_2"] & S_2 \arrow[loop above, "p_2"] \arrow[dl, "a_2"] \\
& S' \arrow[ul, hook, "a_1"] \arrow[loop below, "p'"] &
\end{tikzcd}$$

where x_1, y_2, a_1 are all of full rank. Then I_i is the subvariety cut out by the relations:

$$a_1 p' = p_1 a_1, \quad a_2 p_2 = p' a_2. \tag{5.5}$$

As a_1 is an embedding, the first equation in (5.5) is equivalent to $p' = p_1|_{S'}$. The map $\pi_i : I_i \to I_0 \subset E_+ \times E_-$ is given by composing $a_1 a_2$ and forgetting p'.

The differentials d in (5.2) are transported via Knörrer periodicity to morphisms of matrix factorizations, which we continue to denote by d. The result is a complex of matrix factorizations:[7]

$$(\pi_k)_* \mathcal{O}_{I_k}[-k^2 - k] \xrightarrow{d} \cdots \xrightarrow{d} (\pi_1)_* \mathcal{O}_{I_1}[-2] \xrightarrow{d} (\pi_0)_* \mathcal{O}_{I_0} \tag{5.6}$$

The totalization of this complex is (up to a line bundle) the kernel of the equivalence $\Psi \Phi_{\mathcal{T}} \Psi^{-1}$.

Theorem 5.7 ([T, Thm. 3.3])**.** *The totalization of* (5.6) *is the kernel for the window equivalence corresponding to the window:*

$$\left\{ \mathbb{S}^{\gamma} S,\ 0 \leq \gamma_k \leq \cdots \leq \gamma_1 < n \right\}$$

Note that for $k = 2$ this is the window $\mathfrak{W}'$ (4.4). Hence by Theorem 1.1:

Corollary 5.8. *When $k = 2$ the totalization of (5.6) is equivalent to $\mathcal{O}_{\overline{\Delta}}$ in the category* $\mathrm{MF}(E_+ \times E_-, W_2 - W_1)$.

Our goal for the remainder of this section is to unpack Corollary 5.8 explicitly. We will find locally free resolutions of each object in (5.6) and see that when we form the convolution, after a lot of cancellation, we get exactly the resolution we found in Section 4.3.

The right setting for this computation is the total space of the vector bundle:

$$\mathrm{End}(S_1) \oplus \mathrm{End}(S_2) \longrightarrow \widetilde{F}$$

This is essentially the space $\mathcal{D}$ from Section 4.1.[8] It carries the superpotential

$$\widetilde{W} = \mathrm{tr}(y_2 x_1 a p_2 - a y_2 x_1 p_1)$$

which is the restriction of $W_2 - W_1$ from $E_+ \times E_-$.

Remark 5.9. If the equations (5.1) cutting out the fibre product $X_+ \times_{X_0} X_-$ in $\widetilde{F}$ were transverse then Knörrer periodicity would give an equivalence:

$$\mathrm{MF}\big(\mathrm{End}(S_1) \oplus \mathrm{End}(S_2)_{\widetilde{F}},\ \widetilde{W}\big) \ \cong\ D^b(X_+ \times_{X_0} X_-)$$

But they are far from transverse. So working in this category of matrix factorizations is the same as equipping the fibre product with a derived structure which is non-trivial at all points.

[7] We omit the common line bundle $(\det S_1)^k \otimes (\det S_2)^{-k}$ that appears in each of the kernels (5.4).
[8] To be precise, it's the open subset of $\mathcal{D}$ where $\mathrm{rank}(y_2) = 2$.

Within $\widetilde{W}=0$ we have the subvariety

$$I_0 = \{\, ap_2 = p_1 a \,\}$$

as defined above. This is similar to the fibre product $X_+ \times_{X_0} \times X_-$ in that it consists of three irreducible components distinguished by the rank of a. One component is the space

$$I_2 = \{\, a = 0 \,\} \quad \subset I_0$$

where $\mathrm{rank}(a)=0$. The component for $\mathrm{rank}(a)=2$ is

$$\overline{\Delta} = \{\, ap_2 = p_1 a,\ \det(p_1)=\det(p_2),\ \mathrm{tr}(p_1)=\mathrm{tr}(p_2) \,\} \quad \subset I_0$$

which equals the fibre product $E_+ \times_{E_0} E_-$ (Section 4.1). Finally we have the space

$$\begin{aligned} I_1 = &\{\, p_1 \text{ preserves the line } S' \subset S_1,\ \ a_2 p_2 = (p_1|_{S'})a_2 \,\} \\ &\subset \ \ \mathrm{End}(S_1)\oplus \mathrm{End}(S_2) \oplus \mathrm{Hom}(S_2, S')_{\mathrm{Fl}(1,2,V)\times \mathrm{Gr}(V,2)} \end{aligned} \tag{5.10}$$

There is a map $\pi_1 : I_1 \to I_0$ whose image consists of two irreducible components: I_2 and the component for $\mathrm{rank}(a)=1$. Setting $k=2$ in (5.6), the Cautis-Kamnitzer-Licata kernel is—after Knörrer periodicity—the totalization of:

$$\mathcal{O}_{I_2}[-6] \longrightarrow (\pi_1)_*\mathcal{O}_{I_1}[-2] \longrightarrow \mathcal{O}_{I_0} \tag{5.11}$$

Now we compute some locally free resolutions. Firstly the locus I_2 is cut out by the transverse section $a \in \Gamma(\mathrm{Hom}(S_2,S_1))$ so we have a Koszul resolution:

$$\mathcal{O}(-2,2) \longrightarrow S_1^\vee \otimes S_2(-1,1) \longrightarrow \begin{array}{c}\scriptstyle \mathrm{Sym}^2 S_2(-1,0)\\ \scriptstyle\oplus \\ \scriptstyle \mathrm{Sym}^2 S_1^\vee(0,1)\end{array} \longrightarrow S_1^\vee \otimes S_2 \longrightarrow \mathcal{O} \tag{5.12}$$

Here for legibility we've adopted the notation $\mathcal{O}(a,b) = (\det S_1)^a(\det S_2)^b$. As usual we can turn this into a 'Koszul-type' matrix factorization by adding the arrows $p_2y_2x_1 - y_2x_1p_1$ in the opposite direction.

Next consider the locus I_0 itself, which is cut out transversely by $ap_2 - p_1 a \in \Gamma(\mathrm{Hom}(S_2,S_1))$. It is equivalent to a Koszul-type matrix factorization:

$$\mathcal{O} \rightleftarrows S_1^\vee \otimes S_2 \rightleftarrows \begin{array}{c}\scriptstyle \mathrm{Sym}^2 S_2(-1,0)\\ \scriptstyle\oplus \\ \scriptstyle \mathrm{Sym}^2 S_1^\vee(0,1)\end{array} \rightleftarrows S_1^\vee \otimes S_2(-1,1) \rightleftarrows \mathcal{O}(-2,2) \tag{5.13}$$

Note that the Koszul resolution of $\mathcal{O}_{I_0}$ consists of the right-to-left arrows only. We're writing it this way around because the section $ap_2 - p_1 a$ has R-charge 2, see Remark 4.11.

This resolves the first and third terms of (5.11). The middle term $(\pi_1)_*\mathcal{O}_{I_1}$ is more complicated. Within the space (5.10), the locus I_1 is cut out by two transverse conditions, so we can resolve $\mathcal{O}_{I_1}$ on this space by the Koszul complex for the two sections

$$p_1 \in \Gamma(\mathrm{Hom}(S', S_1/S')) \qquad \text{and} \qquad a_2p_2 - p_1a_2 \in \Gamma(\mathrm{Hom}(S_2, S'))$$

which both have R-charge 1. Also, we have an embedding

$$\mathrm{Hom}(S_2,S')_{\mathrm{Fl}(1,2,V)\times\mathrm{Gr}(V,2)} \hookrightarrow \mathrm{Hom}(S_2,S_1)_{\mathrm{Fl}(1,2,V)\times\mathrm{Gr}(V,2)}$$

whose image is cut out by the transverse section

$$a_2 \in \Gamma(\mathrm{Hom}(S_2, S_1/S'))$$

(which has R-charge zero). Using these three Koszul complexes, and pushing-down along $\mathrm{Fl}(1,2,V) \to \mathrm{Gr}(2,V)$, we construct a locally free resolution of $(\pi_1)_*\mathcal{O}_{I_1}$

which has underlying $\mathbb{C}^*$-equivariant vector bundle:

$$
\begin{array}{ccccc}
\mathcal{O}(-1,1) &
\begin{matrix} \mathcal{O} \\ \oplus \\ \mathrm{Sym}^2 S_2(-1,0) \\ \oplus \\ \mathrm{Sym}^2 S_1^\vee(0,1) \\ \oplus \\ \mathcal{O}(-1,1) \\ \oplus \\ \mathcal{O}(-2,2) \end{matrix} &
\begin{matrix} S_1^\vee \otimes S_2 \\ \oplus \\ S_1^\vee \otimes S_2 \\ \oplus \\ S_1^\vee \otimes S_2(-1,1) \\ \oplus \\ S_1^\vee \otimes S_2(-1,1) \end{matrix} &
\begin{matrix} \mathcal{O} \\ \oplus \\ \mathrm{Sym}^2 S_2(-1,0) \\ \oplus \\ \mathrm{Sym}^2 S_1^\vee(0,1) \\ \oplus \\ \mathcal{O}(-1,1) \\ \oplus \\ \mathcal{O}(-2,2) \end{matrix} &
\mathcal{O}(-1,1)
\end{array}
\tag{5.14}
$$

We have not attempted to indicate the differentials in this resolution. Once again one can turn it into a matrix factorization by adding further arrows.

Now we wish to compare the convolution of (5.12), (5.13) and (5.14) with $\mathcal{O}_{\overline{\Delta}}$. In Section 4.3 we ignored traces, so to get a resolution of $\mathcal{O}_{\overline{\Delta}}$ we must take two copies of (4.12) and form the cone on the map $\mathrm{tr}(p_2) - \mathrm{tr}(p_1)$, which has R-charge 2. The result is a matrix factorization with underlying $\mathbb{C}^*$-equivariant vector bundle:

$$
\begin{array}{ccccc}
\begin{matrix} \mathcal{O} \\ \oplus \\ \mathcal{O}(-1,1) \end{matrix} &
\begin{matrix} S_1^\vee \otimes S_2 \\ \oplus \\ \mathcal{O} \\ \oplus \\ \mathcal{O}(-1,1) \end{matrix} &
\begin{matrix} S_1^\vee \otimes S_2 \\ \oplus \\ S_1^\vee \otimes S_2 \end{matrix} &
\begin{matrix} \mathcal{O} \\ \oplus \\ \mathcal{O}(-1,1) \\ \oplus \\ S_1^\vee \otimes S_2 \end{matrix} &
\begin{matrix} \mathcal{O} \\ \oplus \\ \mathcal{O}(-1,1) \end{matrix}
\end{array}
$$

This agrees with the convolution (5.11) assuming that the blue and red terms cancel as indicated:

$$
\begin{array}{ccccc}
{\color{blue}\mathcal{O}(-2,2)} &
{\color{blue}S_1^\vee \otimes S_2(-1,1)} &
\begin{matrix} {\color{blue}\mathrm{Sym}^2 S_2(-1,0)} \\ {\color{blue}\oplus} \\ {\color{blue}\mathrm{Sym}^2 S_1^\vee(0,1)} \end{matrix} &
S_1^\vee \otimes S_2 &
\mathcal{O}
\\[2em]
\mathcal{O}(-1,1) &
\begin{matrix} \mathcal{O} \\ \oplus \\ {\color{red}\mathrm{Sym}^2 S_2(-1,0)} \\ {\color{red}\oplus} \\ {\color{red}\mathrm{Sym}^2 S_1^\vee(0,1)} \\ \oplus \\ \mathcal{O}(-1,1) \\ \oplus \\ {\color{blue}\mathcal{O}(-2,2)} \end{matrix} &
\begin{matrix} S_1^\vee \otimes S_2 \\ \oplus \\ S_1^\vee \otimes S_2 \\ \oplus \\ {\color{blue}S_1^\vee \otimes S_2(-1,1)} \\ \oplus \\ {\color{red}S_1^\vee \otimes S_2(-1,1)} \end{matrix} &
\begin{matrix} \mathcal{O} \\ \oplus \\ {\color{blue}\mathrm{Sym}^2 S_2(-1,0)} \\ {\color{blue}\oplus} \\ {\color{blue}\mathrm{Sym}^2 S_1^\vee(0,1)} \\ \oplus \\ \mathcal{O}(-1,1) \\ \oplus \\ {\color{red}\mathcal{O}(-2,2)} \end{matrix} &
\mathcal{O}(-1,1)
\\[2em]
\mathcal{O} &
S_1^\vee \otimes S_2 &
\begin{matrix} {\color{red}\mathrm{Sym}^2 S_2(-1,0)} \\ {\color{red}\oplus} \\ {\color{red}\mathrm{Sym}^2 S_1^\vee(0,1)} \end{matrix} &
{\color{red}S_1^\vee \otimes S_2(-1,1)} &
{\color{red}\mathcal{O}(-2,2)}
\end{array}
$$

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